Nonparametric Identification of First-Price Auction Models with Unbounded Values and Observed Auction-Specific Heterogeneity

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Abstract

In all cases in the nonparametric auction literature it is assumed that the value distribution has known bounded support. In this paper we show via an alternative nonparametric identification proof that this assumption is superfluous, provided that the value distribution has a finite expectation. In first instance we show this for the case of repeated identical first price auctions, and then we extend the proof to the case of first price auctions with observed auction-specific heterogeneity. Also, we consider the case where the log of the values is modeled as a median regression model, and the case where the bidders know ex-ante the actual number of bidders rather than the number of potential bidders.

Key words: First price auctions, auction-specific heterogeneity, nonparametric identification

JEL codes: C14, D44

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1 Introduction

In this paper we show the nonparametric identification of first-price sealed bid auction models under mild conditions, where the values of the potential bidders are independent and private and bidders are ex-ante identical, possibly conditional on observed auction specific covariates. This is known as the Independent Private Values (IPV) paradigm. Moreover, we assume risk neutrality. Furthermore, we assume that after the auction the bids are unsealed, and are therefore ex-post observable. In the sequel we call this type of auctions shortly "first-price auctions". Three situations will be investigated in the paper. The first one is the (not very realistic) case where identical auctions are repeated independently with the same known number of potential bidders. The second is the more realistic case where auction-specific characteristics are observed and the number of potential bidders and the reservation price are allowed to change. The third is the case where the reservation price is binding and the bidders know ex-ante the actual number of bidders rather than the number of potential bidders.

As to the literature, there are two seminal papers on the identification of first-price auction models, namely Donald and Paarsch (1996) and Guerre, Perrigne and Vuong (2000). Of course, parametric identification has been developed earlier. In particular, Laffont, Ossard and Vuong (1995) specify the conditional distribution of the log of the private values as normal with conditional mean a linear function of covariates.

Donald and Paarsch (1996) show the nonparametric identification of first-price auctions under the assumption that the support of the distribution $F(v)$ of the values is a known bounded interval $(\underline{v}, \overline{v})$, i.e., $F(v)$ is absolutely continuous with density f such that $f(v) > 0$ on $(\underline{v}, \overline{v})$, and $F(\underline{v}) = 0$, $F(\overline{v}) = 1$. [1] Given the well-known equilibrium bid function of first-price auctions without binding reservation price (see for example Krishna 2002),

$$b = \beta(v) = v - \frac{1}{F(v)^{I-1}} \int_{\underline{v}}^{v} F(x)^{I-1} dx, \tag{1}$$

where I is the number of potential bidders, this assumption implies that also the bid distribution is bounded, with lower bound $\underline{b} = \beta(\underline{v}) = \underline{v}$ and upper bound $\overline{b} = \beta(\overline{v})$.

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[1] More generally, they use the family of Hara utility functions to model non-neutral risk.

Guerre, Perrigne and Vuong (2000) make the same assumption about the value distribution. However, unlike Donald and Paarsch (1996), these authors use the inverse bid function

$$v = \beta^{-1}(b) = b + \frac{1}{I-1}\frac{\Lambda(b)}{\lambda(b)}$$

where $\Lambda(b)$ is the distribution function of the bids, $\lambda(b)$ is the corresponding density, and I is the number of potential bidders. Since the bids are observable, the bids distribution $\Lambda(b)$ and its density $\lambda(b)$ may be considered given, because they can be estimated nonparametrically. Therefore, the private values can uniquely be recovered from the bids and their distribution.

The nonparametric approach of Guerre, Perrigne and Vuong (2000) has been extended by Athey and Haile (2002, 2006a-b) to more general auction models. See Milgrom and Weber (1982) for the latter. Li, Perrigne and Vuong (2000) have extended the nonparametric approach to the conditionally independent private value (CIPV) model, under the assumption that each private value is the product of an idiosyncratic component and a common component. Li and Perrigne (2003) study first-price auctions with random reservation price and show the nonparametric identification of this model. Campo et al. (2002) consider the case of risk averse bidders.

In all cases in the nonparametric auction literature it is assumed that the value distribution has known bounded support. In this paper we show via an alternative nonparametric identification proof that this assumption is superfluous, provided that the value distribution has a finite expectation. In first instance we show this, in Section 2, for the case of repeated identical first price auctions, and then we extend the proof in Section 3 to the case of first price auctions with observed auction-specific heterogeneity. Also, we consider the case where the log of the values is modeled as a median regression model

The standard assumption of first-price auction models is that the number of potential bidders is ex-ante known to the bidders and ex-post to the econometrician as well. The latter is often not the case in practice if the reservation price is binding. Therefore, in Section 4 we consider the case where the bidders know ex-ante the actual number of bidders, i.e., the number of bidders with a value larger than the reservation price, rather than the number of potential bidders. Finally, in Section 5 we will sketch how we plan to use these results in our further research on semi-nonparametric estimation of the (conditional) value distribution of first-price auctions.

2 Repeated identical first-price auctions

The case where a first-price auction is repeated identically is of limited practical interest, but we will consider this case here to illustrate the main ideas behind our alternative identification proof. The more realistic case of first-price auctions with auction-specific observed heterogeneity will be considered in the next section.

2.1 The bid function

Suppose there are I ex-ante identical bidders and there is an indivisible object to sell. Assume that bidders are risk-neutral. Bidders' values are assumed to be independent and private. Moreover, the bidders' values V follow a distribution $F(v)$. Then, given the seller's reservation price p_0 which is announced in advance, the equilibrium bid of a bidder with value v is[2]

$$\beta(v) = v - \frac{1}{F(v)^{I-1}} \int_{\max(p_0, \underline{v})}^{v} F(x)^{I-1} dx, \ v > \max(p_0, \underline{v}), \qquad (2)$$

where I is the number of potential bidders, which is assumed to be known, and

$$\underline{v} = \inf_{F(v)>0} v \qquad (3)$$

is the lower bound of the support of the private values distribution F. See Riley and Samuelson (1981) or Krishna (2002) for the derivation of (2).

We do not restrict $\underline{v}$ to be positive valued, nor do we assume that $\underline{v}$ is known. If $\underline{v} > 0$ and the seller sets the reserve price p_0 below $\underline{v}$, so that p_0 is non-binding, or if there is no reservation price ($p_0 = 0$), every potential bidder will enter the auction. This case is observable because then the number of bids equals the number I of potential bidders.

On the other hand, if the reservation price p_0 is binding, $p_0 > \underline{v}$, only potential bidders with value $V > p_0$ will enter the auction and make their bids. This case is observable because then the actual number if bids, I^*, is less than the number I of potential bidders.

Note that in the case of a binding reservation price p_0, (2) can be written as

$$\beta(v) \ = \ v - \frac{v - p_0}{F(v)^{I-1}} + \frac{1}{F(v)^{I-1}} \int_{p_0}^{v} \left(1 - F(x)^{I-1}\right) dx$$

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[2]See the Appendix for the derivation of the similar expression (16).

$$= \frac{vF(v)^{I-1} - v + p_0}{F(v)^{I-1}} + \frac{1}{F(v)^{I-1}} \int_{p_0}^{v} \left(1 - F(x)^{I-1}\right) dx$$

$$= \frac{p_0}{F(v)^{I-1}} + \frac{\int_{p_0}^{v} \left(1 - F(x)^{I-1}\right) dx - v\left(1 - F(v)^{I-1}\right)}{F(v)^{I-1}}$$

$$= \frac{\int_{p_0}^{v} \left(1 - F(x)^{I-1} - \frac{d}{dx}\left(x\left(1 - F(x)^{I-1}\right)\right)\right) dx}{F(v)^{I-1}}$$

$$+ \frac{p_0 - p_0(1 - F(p_0))^{I-1}}{F(v)^{I-1}}$$

$$= (I-1)\frac{\int_{p_0}^{v} xF(x)^{I-2}F'(x)dx}{F(v)^{I-1}} + p_0\frac{F(p_0)^{I-1}}{F(v)^{I-1}}.$$

Consequently,

$$\lim_{v \to \infty} \beta(v) = (I-1)\int_{p_0}^{\infty} xF(x)^{I-2}F'(x)dx + p_0 F(p_0)^{I-1} < \infty \qquad (4)$$

if and only if

Assumption 1. *The value distribution has a finite expected value: $E[V] < \infty$.*

The importance of (4) is that then the expected revenue of the seller, $\int_{p_0}^{\infty} \beta(v) F'(v)dv$, is then finite too.

2.2 Non-binding reservation price

In an auction with a non-binding reservation price, we may without loss of generality assume that the seller sets $p_0 = 0$ so that the bid function (2) becomes (1). The problem is that this bid function depends on $\underline{v}$, which is unknown. However, if we replace the nonrandom argument v in (1) with a random drawing V from $F(v)$ we do not need to bother about $\underline{v}$, because then

$$\beta(V) = V - \frac{1}{F(V)^{I-1}} \int_{\underline{v}}^{V} F(x)^{I-1}dx = V - \frac{1}{F(V)^{I-1}} \int_{0}^{V} F(x)^{I-1}dx$$

a.s.[3], due to the fact $P[V > \underline{v}] = 1$ and thus $P[F(V) > 0] = 1$

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[3]a.s. stands for almost surely, or equivalently, with probability 1.

Now suppose that there exist two distinct value distribution $F_*(v)$ different from $F(v)$ such that, with V a random drawing from $F(v)$ and V_* a random drawing from $F_*(v)$, such that

$$\beta_*(V_*) = V_* - \frac{1}{F_*(V_*)^{I-1}} \int_0^{V_*} F_*(x)^{I-1} dx$$

has the same distribution as $\beta(V)$. In other words, $F(v)$ and $F_*(v)$ are **observationally equivalent** (see Roehrig 1988). We will show that if $F(v)$ and $F_*(v)$ are observationally equivalent then they are identical: $F(v) = F_*(v)$ on $(0, \infty)$, provided that both distributions are absolutely continuous with connected support:

Assumption 2. *In a first-price sealed bid auction, the value distribution is confined to the class $\mathcal{F}_{accs}$ of absolutely continuous distributions with connected supports.*

Connectedness of the support of $F(v)$ means that the support $\{v \in (0, \infty) : F'(v) > 0\}$ takes the form of an interval.

Note that we do not assume that the supports of $F(v)$ and $F_*(v)$ are equal, but only that they are connected.

The main reason for this assumption is the following well-known result, which follows trivially from the fact that each F is strictly monotonic and therefore invertible on its support.

Lemma 1. *Let V be a random drawing from an absolutely continuous distribution F with connected support. Then $U = F(V)$ has a uniform $[0, 1]$ distribution, and there exists an inverse function F^{-1} on $[0, 1]$ such that $V = F^{-1}(U)$ a.s.*

Under Assumption 2 it follows from Lemma 1 that $U = F(V)$ and $U_* = F_*(V_*)$ are uniformly $[0, 1]$ distributed, so that

$$B = \varphi(U) = F^{-1}(U) - \frac{1}{U^{I-1}} \int_0^{F^{-1}(U_1)} F(x)^{I-1} dx$$

and

$$B_* = \varphi_*(U_*) = F_*^{-1}(U_*) - \frac{1}{U_*^{I-1}} \int_0^{F_*^{-1}(U_*)} F_*(x)^{I-1} dx$$

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have the same distribution:

$$P[B \leq b] = P[B_* \leq b] = \Lambda(b), \tag{5}$$

say.

Since $\varphi(u)$ is monotonic increasing and therefore invertible on $(0,1)$, it follows from (5) that for all b in the support of $\Lambda(b)$,

$$
\begin{aligned}
\varphi_1^{-1}(b) &= P[U_1 \leq \varphi_1^{-1}(b)] = P[\varphi_1(U_1) \leq b] \\
&= P[B \leq b] = P[B_* \leq b] \\
&= P[\varphi_*(U_*) \leq b] = P[U_* \leq \varphi_*^{-1}(b)] = \varphi_*^{-1}(b).
\end{aligned}
$$

Hence, $\varphi_1(u) = \varphi_2(u)$ a.e.[4] on $(0,1)$ and thus by continuity,

$$F^{-1}(u) - \frac{1}{u^{I-1}} \int_0^{F^{-1}(u)} F(x)^{I-1} dx = \varphi(u)$$

$$= \varphi_*(u) = F_*^{-1}(u) - \frac{1}{u^{I-1}} \int_0^{F_*^{-1}(u)} F_*(x)^{I-1} dx$$

exactly on $(0,1)$. Multiplying both sides of this equation by u^{I-1} yields

$$u^{I-1} F^{-1}(u) - \int_0^{F^{-1}(u)} F(x)^{I-1} dx = u^{I-1} F_*^{-1}(u) - \int_0^{F_*^{-1}(u)} F_*(x)^{I-1} dx$$

and then taking the derivative to $u \in (0,1)$ yields

$$(I-1)u^{I-2} F^{-1}(u) + u^{I-1} \frac{dF^{-1}(u)}{du} - (F(F^{-1}(u)))^{I-1} \frac{dF^{-1}(u)}{du}$$

$$= (I-1)u^{I-2} F_*^{-1}(u) + u^{I-1} \frac{dF_*^{-1}(u)}{du} - (F_*(F_*^{-1}(u)))^{I-1} \frac{dF_*^{-1}(u)}{du},$$

so that $F^{-1}(u) = F_*^{-1}(u)$ for all $u \in (0,1)$. Consequently, $F(v)$ and $F_*(v)$ are equal on a common support and therefore $F(v) = F_*(v)$ on $[0, \infty)$.

2.3 Binding reservation price

In the binding reservation price case some bidders' values are above p_0 while some bidders' values are below p_0. The former bidders submit their bids according to the equilibrium bid function

$$\beta(v) = v - \frac{1}{F(v)^{I-1}} \int_{p_0}^v F(x)^{I-1} dx, \quad v > p_0, \tag{6}$$

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[4]a.e. stands for almost everywhere, which means that the property holds except perhaps on a set with zero Lebesgue measure.

whereas the latter bidders do not submit the bid. In the latter case we may assume without loss of generality that these potential bidders submit zero bids. After the auction, the econometrician can observe the number actual bids, I^*, and the number $I - I^*$ of zero bids. The number $I - I^*$ has a $\text{Bin}(I, F(p_0))$ distribution, hence $E\left[(I - I^*)/I\right] = F(p_0)$. In L repeated identical auctions, where for each action ℓ the actual number of bidders is I_ℓ^*, $F(p_0)$ can be estimated consistently by $(1/L)\sum_{j=1}^{L}(I - I_\ell^*)/I$. Therefore,

$$\alpha = F(p_0)$$

is nonparametrically identified and may be taken as given.

Now consider the conditional distribution

$$
\begin{aligned}
\underline{F}(v) &= P[V \le v | V > p_0] = \frac{P[p_0 < V \le v]}{P[V > p_0]} \qquad (7) \\
&= \frac{F(v) - F(p_0)}{1 - F(p_0)} = \frac{F(v) - \alpha}{1 - \alpha} \text{ if } v \ge p_0, \\
\underline{F}(v) &= 0 \text{ if } v < p_0.
\end{aligned}
$$

Then

$$F(v) = \alpha + (1 - \alpha)\underline{F}(v). \qquad (8)$$

Substituting (8) in (6) yields

$$
\begin{aligned}
\beta(v) &= F^{-1}\left(\alpha + (1 - \alpha)\underline{F}(v)\right) - \frac{1}{\left(\alpha + (1 - \alpha)\underline{F}(v)\right)^{I-1}} \\
&\quad \times \int_{p_0}^{F^{-1}(\alpha + (1-\alpha)\underline{F}(v))} \left(\alpha + (1 - \alpha)\underline{F}(x)\right)^{I-1} dx, \ v > p_0.
\end{aligned}
$$

Given that F satisfies Assumption 2, it follows that $\underline{F}$ also satisfies the conditions in Assumption 2, hence $\underline{F}$ is invertible on its support, with inverse denoted by $\underline{F}^{-1}(.)$. It follows therefore from Lemma 1 that for a random drawing $\underline{V}$ from $\underline{F}$, $U = \underline{F}(\underline{V})$ has a uniform $[0, 1]$ distribution, and hence the bids B, including the zero bids, are distributed according to

$$
\begin{aligned}
B &\sim \left(F^{-1}\left(\alpha + (1 - \alpha)U\right) - \frac{1}{\left(\alpha + (1 - \alpha)U\right)^{I-1}} \right. \qquad (9) \\
&\quad \left. \times \int_{p_0}^{F^{-1}(\alpha + (1-\alpha)U)} \left(\alpha + (1 - \alpha)\underline{F}(x)\right)^{I-1} dx \right).D
\end{aligned}
$$

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where U is distributed uniform $[0, 1]$, and

$$D = \mathbf{1}\left(V > p_0\right), V \sim F(v) \tag{10}$$

where $\mathbf{1}\left(.\right)$ is the indicator function[5], with distribution $P[D = 0] = \alpha$, $P[D = 1] = 1 - \alpha$. Since U was actually drawn conditionally on the event $V > p_0$, it follows that U and D are independent.

Suppose there exists a distribution $F_*\left(v\right)$ with $F_*\left(p_0\right) = \alpha$ and corresponding conditional distribution function

$$\underline{F}_*(v) = \frac{F_*\left(v\right) - \alpha}{1 - \alpha} \text{ if } v \geq p_0, \ \underline{F}_*(v) = 0 \text{ if } v < p_0$$

such that

$$\begin{aligned} B \ \sim \ & \left(F_*^{-1}\left(\alpha + \left(1 - \alpha\right) U_*\right) - \frac{1}{\left(\alpha + \left(1 - \alpha\right) U_*\right)^{I-1}} \right. \\ & \left. \times \int_{p_0}^{F_*^{-1}\left(\alpha + \left(1 - \alpha\right) U_*\right)} \left(\alpha + \left(1 - \alpha\right) \underline{F}_*(x)\right)^{I-1} dx \right).D_*, \end{aligned} \tag{11}$$

where U_* is uniformly $[0, 1]$ distributed, and $D_* = 1\left(V > p_0\right)$, $V \sim F_*(v)$, with the same distribution as (10). Again, U_* and D_* are independent. Since D and D_* have the same distribution, it suffices to compare the right-hand sides of (9) and (11) conditional on $D = 1$ and $D_* = 1$, respectively. Then similar to the non-binding reservation price case we must have that for all $u \in (0, 1)$,

$$\begin{aligned} & F^{-1}\left(\alpha + \left(1 - \alpha\right) u\right) \\ & - \frac{1}{\left(\alpha + \left(1 - \alpha\right) u\right)^{I-1}} \int_{p_0}^{F^{-1}\left(\alpha + \left(1 - \alpha\right) u\right)} \left(\alpha + \left(1 - \alpha\right) \underline{F}(x)\right)^{I-1} dx \\ & = F_*^{-1}\left(\alpha + \left(1 - \alpha\right) u\right) \\ & \quad - \frac{1}{\left(\alpha + \left(1 - \alpha\right) u\right)^{I-1}} \int_{p_0}^{F_*^{-1}\left(\alpha + \left(1 - \alpha\right) u\right)} \left(\alpha + \left(1 - \alpha\right) \underline{F}_*(x)\right)^{I-1} dx, \end{aligned}$$

hence, by change of variables, for all $u \in (\alpha, 1)$,

$$\begin{aligned} & u^{I-1} F^{-1}\left(u\right) - \int_{p_0}^{F^{-1}(u)} \left(\alpha + \left(1 - \alpha\right) \underline{F}(x)\right)^{I-1} dx \\ & \qquad = u^{I-1} F_*^{-1}\left(u\right) - \int_{p_0}^{F_*^{-1}(u)} \left(\alpha + \left(1 - \alpha\right) \underline{F}_*(x)\right)^{I-1} dx \ . \end{aligned}$$

---

[5] $\mathbf{1}\left(true\right) = 1$, $\mathbf{1}\left(false\right) = 0$.

Taking the derivative to $u \in (\alpha, 1)$ it follows that

$$(I - 1) u^{I-2} F^{-1}(u) + u^{I-1} \frac{dF^{-1}(u)}{du}$$
$$- \left(\alpha + (1 - \alpha) \underline{F} \left(F^{-1}(u) \right) \right)^{I-1} \frac{dF^{-1}(u)}{du}$$
$$= (I - 1) u^{I-2} F_*^{-1}(u) + u^{I-1} \frac{dF_*^{-1}(u)}{du}$$
$$- \left(\alpha + (1 - \alpha) \underline{F}_* \left(F_*^{-1}(u) \right) \right)^{I-1} \frac{dF_*^{-1}(u)}{du},$$

hence $F^{-1}(u) = F_*^{-1}(u)$ on $(\alpha, 1)$ and thus $F(v) = F_*(v)$ on $[p_0, \infty)$.

3 First-price auctions with observed auction-specific heterogeneity

Let X be the vector of auction-specific characteristics for an auctioned item, with support S_X. The number of potential bidders of an auction with characteristics $X = x \in S_X$ is a known function $I(x)$ of x, but we maintain the assumption that ex-ante $I(x)$ is known to the potential bidders and ex-post to the econometrician. The same applies to the reservation price $p_0(x)$. The conditional value distribution in each auction with characteristics $X = x \in S_X$ is denoted by

$$F(v|x) = P[V \le v | X = x],$$

which is known to each potential bidder. The values themselves are independent within and across auctions, conditional on X.

Since the nonbinding reservation price case follows directly from the binding case by setting $p_0 = 0$, we will focus only on the binding reservation price case. In that case the conditional equilibrium bid function for the actual bids is

$$\beta(v|X) = v - \frac{1}{F(v|X)^{I(X)-1}} \int_{p_0(X)}^{v} F(y|X)^{I(X)-1} dy, \ \ v > p_0(X).$$

Note that Assumption 1 implies that $E[V|X] < \infty$ a.s., so that under Assumption 1, $\lim_{v \to \infty} \beta(v|X) < \infty$ a.s.

3.1   Nonparametric identification

In each auction with characteristics X and reservation price $p_0(X)$ the number of potential bidders $I(X)$ minus the number of actual bidders $I_*(X)$ has a conditional $\mathrm{Bin}(I(X), F(p_0(X)|X))$ distribution, hence

$$E\left[\left.\frac{I(X) - I_*(X)}{I(X)}\right| X\right] = F(p_0(X)|X)$$

which can be consistently estimated by nonparametric kernel regression, given a random sample of auctions. Therefore,

$$\alpha(X) = F(p_0(X)|X)$$

is nonparametrically identified and may be taken as given. Interpreting the non-bids as zero bids, the bids in this auction are distributed as

$$B \sim \left(V - \frac{1}{F(V|X)^{I(X)-1}} \int_{p_0(X)}^{V} F(y|X)^{I(X)-1} dy\right) \mathbf{1}\left(V > p_0(X)\right),$$

where $\mathbf{1}(.)$ is the indicator function.

   Similar to (7), let

$$\underline{F}(v|X) = \frac{F(v|X) - \alpha(x)}{1 - \alpha(x)} \text{ if } v \geq p_0(X), \ \underline{F}(v|X) = 0 \text{ if } v < p_0(X)$$

so that

$$F(v|X) = \alpha(X) + (1 - \alpha(X))\underline{F}(v|X). \tag{12}$$

Moreover, let $\underline{V}$ be a random drawing from $\underline{F}(v|X)$, conditional on X, and let $U = \underline{F}(\underline{V}|X)$. In order to conclude that U has a uniform $[0,1]$ distribution we need to generalize that Assumption 2 to:

Assumption 3. *In a first-price sealed bid auction with auction-specific covariates X, the conditional value distribution given X is confined to the class $\mathcal{F}_{accs}(X)$ of absolutely continuous conditional distributions with connected supports.*

Note that in this case the endpoints of the support may be (Borel measurable) functions of X.

   Now Lemma 1 can be generalized to:

Lemma 2. *Conditional on X, let V be a random drawing from a conditional distribution $F(.|X) \in \mathcal{F}_{accs}(X)$. Then $U = F(V|X)$ has a uniform $[0,1]$ distribution, and U and X are independent. Moreover, for each point x in the support of X there exists an inverse function $F^{-1}(.|x)$ on $[0,1]$ such that $V = F^{-1}(U|X)$ a.s.*

Proof: Appendix.

Similar to (9) we now have that the conditional distribution of the bids (including the zero bids) is

$$
\begin{aligned}
B|X \sim \; & \left(F^{-1}(\alpha(X) + (1-\alpha(X))U|X) - \frac{1}{U^{I(X)-1}} \right. \\
& \left. \times \int_{p_0(X)}^{F^{-1}(\alpha(X)+(1-\alpha(X))U|X)} F(y|X)^{I(X)-1} dy \right).D,
\end{aligned}
$$

where U is uniformly $[0,1]$ distributed, independently of X, and $D = \mathbf{1}(V > p_0(X))$. Note that U is independent of D as well, because U was actually drawn conditionally on X **and** the event $V > p_0(X)$. Now by the same argument as in the case without covariates it follows straightforwardly that conditional on X, $F(v|X)$ is nonparametrically identified on $[p_0(X), \infty)$.

3.2   Semi-nonparametric identification

In order to put some structure on $F(v|X)$, we will now assume that

$$
\ln V = \gamma(X) + \varepsilon, \tag{13}
$$

where

Assumption 4. *The random variable ε in (13) is independent of X, and its distribution is absolutely continuous with connected support.*

The reason for considering this case will be given at the end of this subsection.

To pin down the location of $\gamma(X)$ we will impose a quantile restriction on the distribution of ε, for example that the median of ε is zero. Moreover, to ensure that $E[V|X] = \exp(\gamma(X)) E[\exp(\varepsilon)] < \infty$ we need to require that $E[\exp(\varepsilon)] < \infty$:

Assumption 5. *The median of ε in (13) is zero:$P(\varepsilon \leq 0) = 1/2$, and $E[\exp(\varepsilon)] < \infty$.*

Thus $\gamma(X)$ is now the conditional median of $\ln V$.

It follows from (13) that

$$
\begin{aligned}
F(v|X) &= P[V \leq v|X] = P[\exp(\varepsilon) \leq v \exp(-\gamma(X))|X] \qquad (14) \\
&= P[\exp(-\exp(\varepsilon)) \geq \exp(-v\exp(-\gamma(X)))|X] \\
&= P[1 - \exp(-\exp(\varepsilon)) \leq 1 - \exp(-v\exp(-\gamma(X)))|X] \\
&= H(1 - \exp(-v\exp(-\gamma(X)))),
\end{aligned}
$$

where

$$
\begin{aligned}
H(u) &= P[1 - \exp(-\exp(\varepsilon)) \leq u] = P[\exp(-\exp(\varepsilon)) \geq 1 - u] \\
&= P[\varepsilon \leq \ln(\ln(1/(1-u)))]
\end{aligned}
$$

which is a distribution function on $(0,1)$. Note that $H(u)$ satisfies the quantile restriction

$$
H\left(1 - e^{-1}\right) = 1/2. \qquad (15)
$$

The question now arises whether $\gamma(X)$ and $H(u)$ are nonparametrically identified. It suffices to establish the uniqueness of $\gamma(X)$ only, because $F(v|X)$ is nonparametrically identified on $[p_0(X), \infty)$, so that given $\gamma(X)$, $H(u)$ is identified on $[1 - \exp(-p_0(X)\exp(-\gamma(X))), 1]$.

To answer this question, note that Assumption 5 implies that $H(u)$ is absolutely continuous with connected support, say $(\underline{u}, \overline{u}) \subset [0,1]$. Then it follows from Lemma 1 that H is invertible on $(\underline{u}, \overline{u})$, with inverse H^{-1}. Consequently, it follows from (14) that $1 - \exp(-v\exp(-\gamma(X))) = H^{-1}(F(v|X))$, hence

$$
\begin{aligned}
v &= \exp(\gamma(X))\ln\left(1/\left(1 - H^{-1}(F(v|X))\right)\right) \\
&= \exp(\gamma(X))\ln\left(1/\left(1 - H^{-1}(\alpha(X) + (1 - \alpha(X))\underline{F}(v|X))\right)\right),
\end{aligned}
$$

where the latter follows from (12). Next, let $\underline{V}$ be a random drawing from $\underline{F}(v|X)$. Then it follows from Lemma 2 that $U = \underline{F}(\underline{V}|X)$ is uniformly $[0,1]$ distributed, and is independent of X, hence

$$
\underline{V} = \exp(\gamma(X))\ln\left(1/\left(1 - H^{-1}(\alpha(X) + (1 - \alpha(X))U)\right)\right)
$$

13

Suppose there exists an alternative median function $\gamma_*(X)$ and an alternative distribution function H_* with inverse H_*^{-1} for which

$$\underline{V} = \exp\left(\gamma_*(X)\right)\ln\left(1/\left(1 - H_*^{-1}\left(\alpha(X) + (1 - \alpha(X))U\right)\right)\right)$$

Then for arbitrary $u \in (0,1)$.

$$\exp\left(\gamma_*(X) - \gamma(X)\right) = \frac{\ln\left(1/\left(1 - H^{-1}\left(\alpha(X) + (1 - \alpha(X))u\right)\right)\right)}{\ln\left(1/\left(1 - H_*^{-1}\left(\alpha(X) + (1 - \alpha(X))u\right)\right)\right)}$$

Since the left-hand side of this equation does not depend on u, the derivative of the right-hand side to u is zero, hence

$$\frac{\ln\left(1 - H^{-1}[\alpha(X) + (1 - \alpha(X))u]\right)}{\ln\left(1 - H_*^{-1}[\alpha(X) + (1 - \alpha(X))u]\right)} = C(X),$$

for example, and thus $\gamma_*(X) = \gamma(X) + \ln(C(X))$. But $\gamma_*(X)$ and $\gamma(X)$ are both conditional medians of $\ln V$, which is only possible if $\ln(C(X)) = 0$ a.s.:

$$\gamma_*(X) = \gamma(X).$$

This implies that

$$H_*(u) = H(u) \text{ for } u \in [1 - \exp(-p_0 \exp(-\gamma(x)), 1].$$

The reason for considering the case (13) is that the distribution function $H(u)$ can easily be estimated semi-nonparametrically using orthonormal Legendre polynomials on the unit interval. See Bierens (2006). Given H and a parametric specification of $\gamma(X)$, for example let $\gamma(X)$ be a linear function of X, $F(v|X)$ can be determined via (14). Moreover, the conditional median of the computed function $F(v|X)$ can then be compared with the parametric specification, on the basis of which a test can be developed for the validity of the parametric specification of the median function. This is left for future research.

4 The case where the actual number of bidders is known to the bidders

The nonparametric identification of the first-price auction model with binding reservation price p_0 depends crucially on the assumption that the number

of potential bidders I is known to the bidders as well as to the econometrician. But usually the econometrician only observes the actual number of bids I_*.

To get around that problem, assume that, instead of the number of potential bidders I, the actual number of bidders $I_* \geq 2$ is ex-ante known to all the bidders. Moreover, assume that a binding reservation price p_0 is set in advance by the seller. Then it can be shown[6] that the equilibrium bid function in this case becomes

$$\beta(v) = v - \frac{1}{\underline{F}(v)^{I_*-1}} \int_{p_0}^{v} \underline{F}(x)^{I_*-1} dx, \tag{16}$$

where $\underline{F}(v)$ is defined in (7).

Similarly, in the presence of auction-specific covariates X the conditional equilibrium bid function becomes

$$\beta(v|X) = v - \frac{1}{\underline{F}(v|X)^{I_*(X)-1}} \int_{p_0(X)}^{v} \underline{F}(y|X)^{I_*(X)-1} dy$$

where

$$\underline{F}(v|X) = \frac{F(v|X) - \alpha(X)}{1 - \alpha(X)}$$

with

$$\alpha(X) = F(p_0(X)|X).$$

We will now set forth conditions under which $F(v|X)$ and $\alpha(X)$ are identified.

If we use for $F(v|X)$ the semi-nonparametric specification (14) with parametrized median function $\gamma(X, \theta_0)$, then

$$\begin{aligned} F(v|X) &= H\left(1 - \exp\left(-v \exp\left(-\gamma(X, \theta_0)\right)\right)\right) \\ &= 1 - H_0\left(\exp\left(-v \exp\left(-\gamma(X, \theta_0)\right)\right)\right) \end{aligned}$$

where

$$H_0(u) = 1 - H(1 - u),$$

so that

$$\alpha(X) = 1 - H_0\left(\exp\left(-p_0(X) \exp\left(-\gamma(X, \theta_0)\right)\right)\right).$$

---

[6] See the Appendix.

Note that $H_0(u)$ satisfies the quantile restriction

$$H_0\left(e^{-1}\right) = 1/2. \tag{17}$$

Hence

$$\underline{F}(v|X) = 1 - \frac{H_0\left(\exp\left(-v\exp\left(-\gamma\left(X,\theta_0\right)\right)\right)\right)}{H_0\left(\exp\left(-p_0(X)\exp\left(-\gamma\left(X,\theta_0\right)\right)\right)\right)}, \quad v \geq p_0(X)$$

Since $\underline{F}(v|X)$ is nonparametrically identified on $[p_0(X), \infty)$, it follows that for given θ_0, $H_0(u)$ is identified on $[0, \exp\left(-p_0(X)\exp\left(-\gamma\left(X,\theta_0\right)\right)\right)]$.

Now suppose that there exists a parameter vector $\theta_1 \neq \theta_0$ and a distribution function H_1 on [0,1] such that

$$\underline{F}(v|X) = 1 - \frac{H_1\left(\exp\left(-v\exp\left(-\gamma\left(X,\theta_1\right)\right)\right)\right)}{H_1\left(\exp\left(-p_0(X)\exp\left(-\gamma\left(X,\theta_1\right)\right)\right)\right)}, \quad v \geq p_0(X)$$

Then for all $v \geq p_0(X)$,

$$\frac{H_1\left(\exp\left(-v\exp\left(-\gamma\left(X,\theta_1\right)\right)\right)\right)}{H_0\left(\exp\left(-v\exp\left(-\gamma\left(X,\theta_0\right)\right)\right)\right)} = \frac{H_1\left(\exp\left(-p_0(X)\exp\left(-\gamma\left(X,\theta_1\right)\right)\right)\right)}{H_0\left(\exp\left(-p_0(X)\exp\left(-\gamma\left(X,\theta_0\right)\right)\right)\right)}$$

Since the right-hand side does not depend on v, it follows that the derivative of the left hand side to $v > p_0(X)$ is zero, which implies (after some rearrangements) that

$$\begin{aligned}
\frac{h_1\left(\exp\left(-v\exp\left(-\gamma\left(X,\theta_1\right)\right)\right)\right)}{h_0\left(\exp\left(-v\exp\left(-\gamma\left(X,\theta_0\right)\right)\right)\right)} & \tag{18} \\
\times \exp\left(-v\left(\exp\left(-\gamma\left(X,\theta_1\right)\right) - \exp\left(-\gamma\left(X,\theta_0\right)\right)\right)\right) & \\
= \frac{H_1\left(\exp\left(-v\exp\left(-\gamma\left(X,\theta_1\right)\right)\right)\right)}{H_0\left(\exp\left(-v\exp\left(-\gamma\left(X,\theta_0\right)\right)\right)\right)} \times \frac{\exp\left(-\gamma\left(X,\theta_0\right)\right)}{\exp\left(-\gamma\left(X,\theta_1\right)\right)} & \\
= \frac{H_1\left(\exp\left(-p_0(X)\exp\left(-\gamma\left(X,\theta_1\right)\right)\right)\right)}{H_0\left(\exp\left(-p_0(X)\exp\left(-\gamma\left(X,\theta_0\right)\right)\right)\right)} \times \frac{\exp\left(-\gamma\left(X,\theta_0\right)\right)}{\exp\left(-\gamma\left(X,\theta_1\right)\right)}, &
\end{aligned}$$

where h_1 and h_0 are the densities of H_1 and H_0, respectively.

Next, impose the condition

$$h_1(0) = h_0(0) = 1 \tag{19}$$

16

and take the limit of (18) for $v \to \infty$. Then

$$\lim_{v \to \infty} \exp\left(-v\left(\exp\left(-\gamma\left(X, \theta_1\right)\right) - \exp\left(-\gamma\left(X, \theta_0\right)\right)\right)\right)$$

$$= \begin{cases} \infty & \text{if} \quad \gamma\left(X, \theta_1\right) > \gamma\left(X, \theta_0\right) \\ 0 & \text{if} \quad \gamma\left(X, \theta_1\right) < \gamma\left(X, \theta_0\right) \\ 1 & \text{if} \quad \gamma\left(X, \theta_1\right) = \gamma\left(X, \theta_0\right) \end{cases}$$

$$= \frac{H_1\left(\exp\left(-v\exp\left(-\gamma\left(X, \theta_1\right)\right)\right)\right)}{H_0\left(\exp\left(-v\exp\left(-\gamma\left(X, \theta_0\right)\right)\right)\right)} \times \frac{\exp\left(-\gamma\left(X, \theta_0\right)\right)}{\exp\left(-\gamma\left(X, \theta_1\right)\right)}$$

$$= \frac{H_1\left(\exp\left(-p_0(X)\exp\left(-\gamma\left(X, \theta_1\right)\right)\right)\right)}{H_0\left(\exp\left(-p_0(X)\exp\left(-\gamma\left(X, \theta_0\right)\right)\right)\right)} \times \frac{\exp\left(-\gamma\left(X, \theta_0\right)\right)}{\exp\left(-\gamma\left(X, \theta_1\right)\right)}.$$

Clearly, only the option

$$\gamma\left(X, \theta_1\right) = \gamma\left(X, \theta_0\right) \tag{20}$$

is possible, which implies that

$$H_1(u) = H_0\left(u\right) \text{ on } \left[0, \ \exp\left(-p_0(X)\exp\left(-\gamma\left(X, \theta_0\right)\right)\right)\right]. \tag{21}$$

The condition (19) can be implemented similar to the condition $h_0(1) = 1$ in Bierens (2006) and Bierens and Carvalho (2006). Of course, we should also impose the quantile restriction (17). Moreover, under Assumption 1 in Bierens and Carvalho (2006), (21) implies that $H_1(u) = H_0\left(u\right)$ on $[0, \ 1]$, so that $H_0\left(u\right)$ is identified on $[0, \ 1]$. Finally, we need some obvious regularity conditions on the distribution of X and the functional form of $\gamma\left(x, \theta\right)$ such that (20) implies $\theta_1 = \theta_0$.

5   Concluding remarks

In this paper we have proved the non-parametric and semi-nonparametric identification of various first-price auction models, with and without binding reservation price, without using the usual condition that the value distribution has known bounded support. These results, in particular the results in Sections 3.2 and 4, are the basic for our further research on semi-nonparametric estimation of these models via semi-nonparametric modeling of density and distribution functions on the unit interval, along the lines in Bierens (2006) and Bierens and Carvalho (2006). See also Chen (2006)

for a review of semi-nonparametric modeling and estimation. In particular, we will propose to estimate these models semi-nonparametrically via a simulated integrated conditional moment criterion, similar to the integrated conditional moment test statistic proposed by Bierens (1982) and Bierens and Ploberger (1997). In our case the moment function is the difference between the empirical characteristic functions of the observed bids and the empirical characteristic function of the corresponding simulated bids generated by the equilibrium bid function for a semi-nonparametric specification of the value distribution.

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6 Appendix

6.1 Proof of Lemma 2

Let $(\underline{v}(X), \overline{v}(X))$ (or its closure) be the support of $F(.|X)$. Since $F(v|X)$ is strictly monotonic increasing on $(\underline{v}(X), \overline{v}(X))$, it is invertible: For each x in the support of X and each $u \in (0,1)$ there exists a unique $v \in (\underline{v}(x), \overline{v}(x))$ such that $F(v|x) = u$, hence there exists a conditional distribution function $F^{-1}(u|x)$ on $[0,1]$ such that $F(v|x) = u \in (0,1)$ implies $v = F^{-1}(u|x) \in (\underline{v}(x), \overline{v}(x))$. Then

$$
\begin{aligned}
P[U \leq u|X] &= P[F(V|X) \leq u|X] = P\left[V \leq F^{-1}(u|X)|X\right] \\
&= F(F^{-1}(u|X)|X) = u.
\end{aligned}
$$

Since the right-hand side does not depend on X, U and X are independent, and therefore $P[U \leq u] = u$.

6.2 Proof of (16)

Let $\beta(v)$ be the strictly monotonic increasing equilibrium bid function involved, and let b be the bid of bidder 1, which corresponds to an x such that $b = \beta(x)$. Given the value $V_1 = v$ of bidder 1, the expected value for bidder 1 of the object to be auctioned off is v times the probability that he wins the object. The latter is the case if his bid $\beta(x)$ is the highest bid, which by the monotonicity of $\beta(v)$ is the case if $x > \overline{V}_2 = \max\{V_2, ..., V_{I_*}\}$. The probability of this event, conditional on $\underline{V}_2 = \min\{V_2, ..., V_{I_*}\} > p_0$, is $\underline{G}(x) = \underline{F}(x)^{I_*-1}$, where $\underline{F}(x)$ is defined by (7), hence the expected value is $v\underline{G}(x)$.

Given the bid $b = \beta(x)$, let $p(x)$ be the expected price to pay to the seller. At this point we do not assume yet that $p(x)$ equals $\beta(x)$ times the probability of winning the auction. Then the expected net gain for bidder 1 is $\pi(v, x) = v\underline{G}(x) - p(x)$, which is maximal if x is chosen such that

$$0 = \partial\pi(v, x)/\partial x = v\underline{G}'(x) - p'(x) \tag{22}$$

In order that the bid $b = \beta(x)$ is an equilibrium bid, the solution of (22) must be $x = v$, hence

$$p'(v) = v\underline{G}'(v). \tag{23}$$

Using the conditions $\underline{G}'(v) = 0$, $\pi(v,v) = 0$ for $v < p_0$, and $p(v) = 0$ at $v = p_0$, the solution of the differential equation (23) is $p(v) = \int_{p_0}^v x\underline{G}'(x)dx = v\underline{G}(v) - \int_{p_0}^v \underline{G}(x)dx$. For the equilibrium bid function $\beta(v)$, $p(v)$ is equal to $\beta(v)$ times the probability $\underline{G}(v)$ of winning the auction: $p(v) = \beta(v)\underline{G}(v) = v\underline{G}(v) - \int_{p_0}^v \underline{G}(x)dx$, hence

$$\beta(v) = v - \frac{1}{\underline{G}(v)}\int_{p_0}^v \underline{G}(x)dx = v - \frac{1}{\underline{F}(v)^{I_*-1}}\int_{p_0}^v \underline{F}(x)^{I_*-1}dx.$$